5.3b population growth

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Malthusian Growth

- Rate of growth proportional to population size.
- Let $a = b - d$ be the birth rate minus death rate.

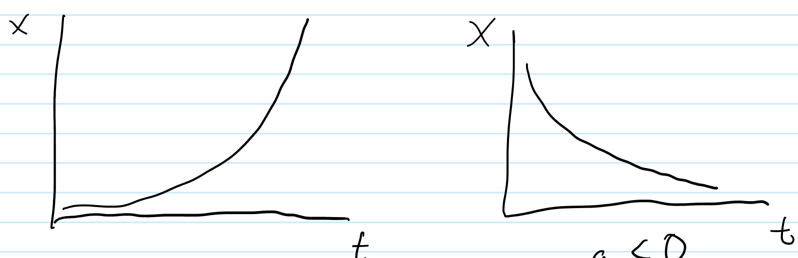
  ↑ proportionality factor

$$\dot{x} = ax = f(x), \qquad a \neq 0.$$

$$\Rightarrow \quad x(t) = x_0 e^{ax}$$

$a\bar{x} = 0 \Rightarrow \bar{x} = 0$ is the only equilibrium.

Note $f'(x) = a$, so $f'(0) = a$, so 0 is locally asymp. stable if $a < 0$
is unstable if $a > 0$.



$a > 0$ $\qquad$ $a < 0$.

Logistic growth (Introduced by Pierre Verhulst, 1837)

Rate of growth $a(x) = r\left(1 - \frac{x}{K}\right)$ ← carrying capacity

- For populations $x > K$, the rate is negative
- For populations $x < K$, the rate is positive.

$$\dot{x} = a(x) \cdot x = r\left(1 - \frac{x}{K}\right)x \quad \leftarrow \text{autonomous nonlinear}$$

$$0 = r\left(1 - \frac{\bar{x}}{K}\right)\bar{x}$$

$$\Rightarrow \bar{x} = 0, K.$$

$$f(x) = r\left(1 - \frac{x}{K}\right)x$$

$$f'(x) = r\left[\left(1 - \frac{x}{K}\right) + x\left(-\frac{1}{K}\right)\right] = r\left(1 - \frac{2x}{K}\right)$$

$$f'(0) = r, \qquad f'(K) = -r$$

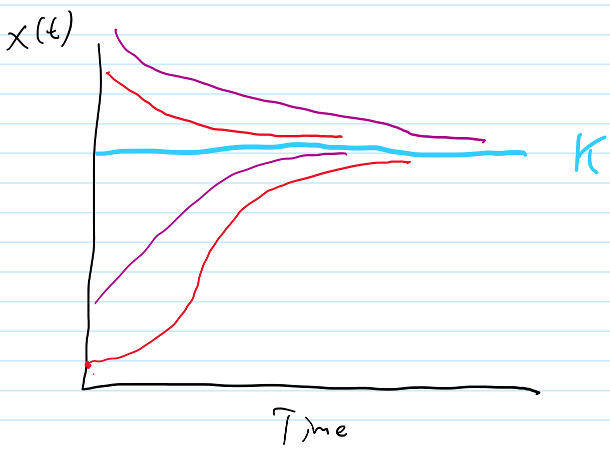
If $r>0$, then 0 is an unstable equilibrium
then K is a locally asymp. stable equilibrium.

If we solve using separation of variables, then

$$x(t) = \frac{x_0 K}{x_0 + (K - x_0)e^{-rt}}, \qquad x_0 = x(0).$$

$$\Rightarrow \lim_{t\to\infty} x(t) = K \quad \text{if} \quad x_0 > 0 \text{ and } r > 0.$$

So K is actually globally asymp. stable.



Allee Effect [1931]

- Often, there's reduced fitness for low population sizes.
- One model: $a(x) = a_1 + a_2 x + a_3 x^2$

reproductive rate

quadratic function

$$\dot{x} = ax = x(a_1 + a_2 x + a_3 x^2), \qquad a_1 < 0, \quad a_2 > 0, \quad a_3 < 0.$$

$$0 = x(a_1 + a_2 x + a_3 x^2)$$
$$0 = a_3 x(x - \alpha_1)(x - \alpha_2), \qquad \text{quadratic equation}$$

$$x = 0, \alpha_1, \alpha_2, \quad \text{and WLOG,} \quad 0 < \alpha_1 < \alpha_2.$$

$$f(x) = a_3 x (x - \alpha_1)(x - \alpha_2)$$
$$f'(x) = a_3 \left[(x - \alpha_1)x + (x - \alpha_2)x + (x - \alpha_1)(x - \alpha_2) \right]$$

$f(x) = a_3 x (x - \alpha_1)(x - \alpha_2)$

$f'(x) = a_3 \left[(x - \alpha_2) x + (x - \alpha_1) x + (x - \alpha_1)(x - \alpha_2) \right]$

$f'(0) = a_3 (-\alpha_1)(-\alpha_2) < 0$, so $\bar{x} = 0$ is locally asymp. stable,

$f'(\alpha_1) = a_3 (\alpha_1 - \alpha_2) \alpha_1 > 0$, so $\bar{x} = \alpha_1$ is unstable,

$f'(\alpha_2) = a_3 (\alpha_2 - \alpha_1) \alpha_2 < 0$, so $\bar{x} = \alpha_2$ is locally asymp. stable.